A General Theory of the Interaction of Electromagnetic Waves with Isotropic, Horizontally Layered Media, and Applications to Propagation Over Sea Ice

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Technical Report
C-CORE Publication No. 82-9
August, 1982

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The Centre for Cold Ocean Resources Engineering (C-CORE) was established at Memorial University of Newfoundland in St. John's in 1975. Its initial funding was provided by the Devonian Group of Charitable Foundations of Calgary. Ongoing funding now comes from government and industry.

The Centre's mandate is to undertake research that will assist the safe and orderly development of Canada's resources in cold and ice frequented oceans.

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ISBN: 088901-085-4

Abstract

A general formulation of the problem of determining the electromagnetic field resulting from a finite electromagnetic source placed above a horizontally layered isotropic medium is presented. A solution to the problem is given in terms of two-dimensional Fourier transforms. The solution has the following properties: the source or exciting field is arbitrary; the resulting electric field intensity is found directly as a function of the source field intensity; boundary conditions evolve directly from the method of solution in the form of auxiliary equations. The general solution is examined for certain special cases and shown to agree in those cases with results available in the literature.

A technique is presented for applying the above results to the problem of electromagnetic propagation predictions over uniform sea ice. This technique was subjected to experimental testing by C-CORE at Pond Inlet in March 1982. Initial analysis of the experimental results shows very good agreement between predicted and measured field strengths. A further complete report in this regard follows in 1983.

Acknowledgement

This work was supported by a C-CORE research contract to the author.

C-CORE acknowledges the continued support of their HF radar program by the Canadian Department of Fisheries and Oceans.

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1. INTRODUCTION

The use of high frequency radar as a remote sensing device for sea ice is being studied at Memorial University. Since the radar would operate in the "ground wave" mode, an effort is being made to understand this type of propagation. The important question for a practical system is one of useable range. Some model is necessary to predict transmission loss.

This report describes certain analytic work undertaken during the period June to December, 1981. The purpose of the work is to develop expressions for electromagnetic fields when they interact with media whose electrical properties change abruptly at various points in one direction. The properties are assumed to be constant in the other directions, i.e. one dimensional layered or stratified and two dimensional isotropic media. This report, of course, is only a step toward developing a model for propagation predictions over an ice infested ocean. Hopefully the results also may be used to predict propagation over uniform sea ice. We will suggest a technique to accomplish this.

The equations suggested for modelling are not new; they are contained in the standard work on this topic by Wait (1970). The analytic approach, however, differs considerably from previously published results. The technique employed is basically the same as that suggested for the treatment of rough surface propagation and scatter (Walsh, 1980). The results obtained here are general, in the sense that the excitation or source remains arbitrary (realizable).

Furthermore, the expressions for the electric field intensities are found directly without the use of Hertz vectors. These two features of the technique lead to formal solutions which, to our knowledge, are more comprehensive than those previously available.

Chapter 2 of this report contains the formulation (obtained directly from Maxwell's equations) and analytic work leading up to the formal general solution. Chapter 3 uses the formal solution from Chapter 2 to develop expressions for the electric field intensity E when the source is a vertical or horizontal electric dipole (the basic antenna element).

Chapter 4 contains a discussion pointing out the relationship between the results of this report and results available in the classical and current literature. The suggested model for the prediction of propagation characteristics over uniform sea ice is also outlined and discussed.

2. THE GENERAL SOLUTION FOR THE ELECTRIC FIELD ABOVE HOROZONTALLY LAYERED MEDIA

2.1 Preliminaries

It is convenient in the following to present detailed analysis first for a two layer model and then at certain points, after results have been obtained, to generalize these results to a multi-layered structure.

Accordingly, our first (two-layer) model is illustrated in Figure 1.

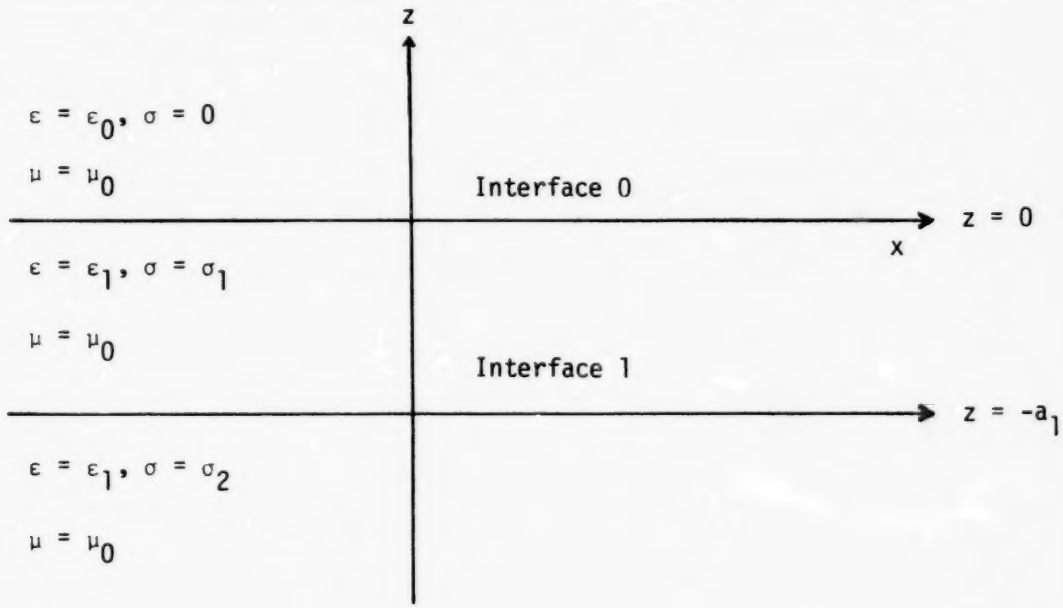

Figure 1 - TWO-LAYER MODEL

The electrical properties of the complete R^3 space may be summarized as,

$$
\begin{array}{rl}
 & \left[\begin{array}{ll}
\varepsilon_0 & \\
0 & 0 < z < \infty \\
\mu_0 &
\end{array}\right. \\
\varepsilon = & \left[\begin{array}{ll}
\varepsilon_1 & \\
\sigma_1 & -a_1 < z < 0 \\
\mu_0 &
\end{array}\right. \\
\sigma = & \\
\mu = & \left[\begin{array}{ll}
\varepsilon_2 & \\
\sigma_2 & -\infty < z < -a_1 \\
\mu_0 &
\end{array}\right.
\end{array}
$$

where ε = the permittivity

μ = the permeability

ε_0 = the permittivity of vacuum

μ_0 = the permeability of vacuum

σ = the electrical conductivity

We will find it convenient to employ the Heaviside functions:

$$h(z) = h_0 = \begin{array}{l} 0, \; z < 0 \\ 1, \; z > 0 \end{array}$$

$$h(z+a_1) = h_1 = \begin{array}{l} 0, \; z < -a_1 \\ 1, \; z > -a_1 \end{array}$$

Using these functions, the electrical properties of the space, may be summarized as:

$$\sigma = (1 - h_1)\,\sigma_2 + (h_1 - h_0)\sigma_1 \tag{1}$$

$$\varepsilon = (1 - h_1)\,\varepsilon_2 + (h_1 - h_0)\varepsilon_1 + h_0\varepsilon_0. \tag{2}$$

2.2 Maxwell's Equations for the Complete Space

We now write Maxwell's equation in their point form, which we assume apply to the complete space:

$$\nabla \times \vec{E} = -j\omega\vec{B} \tag{3}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J} \tag{4}$$

$$\nabla \cdot \vec{B} = 0 \tag{5}$$

$$\nabla \cdot \vec{D} = \rho \tag{6}$$

where the field components have their usual notation. This is the well known "time harmonic form", which we prefer to think of as the Fourier transform, with respect to the time variable of the time-dependent equation. The transformed variable is 'ω' and we use the engineer's notation $\sqrt{(-1)} = j$.

Since the media are assumed to be isotropic, we have

$$\vec{B} = \mu_0 \vec{H} \qquad \text{for all } z \tag{7}$$

$$\vec{D} = \begin{cases} \varepsilon_0\vec{E} & 0 < z < \infty \\ \varepsilon_1\vec{E} & -a_1 < z < 0 \\ \varepsilon_2\vec{E} & -\infty < z < -a_1 \end{cases} \tag{8}$$

$$\vec{J}_c = \begin{cases} 0 & 0 < z < \infty \\ \sigma_1 \vec{E} & -a_1 < z < 0 \\ \sigma_2 \vec{E} & -\infty < z < -a_1 \end{cases} \tag{9}$$

where $\vec{J}_c$ denotes the conduction currents. We may summarize (8) and (9) as:

$$\vec{D} = [(1 - h_1)\varepsilon_2 + (h_1 - h_0)\varepsilon_1 + h_0\varepsilon_0]\,\vec{E} \tag{10}$$

$$\vec{J}_c = [(1 - h_1)\sigma_2 + (h_1 - h_0)\sigma_1]\,\vec{E} \tag{11}$$

We shall interpret all the above equations in the sense of generalized functions. It will be obvious, however, that our method of dealing with these equations implies certain restrictions on the behaviour of the fields and their spatial derivatives immediately above and below each interface. The implication of these restrictions is that we are seeking solutions not in their widest sense, but in a limited class of generalized functions.

2.3 The Basic Equation for the Electrical Field

Consider (4), which we rewrite in the following form:

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}_c + \vec{J}_s$$

$$= j\omega\,[\vec{D} + 1/j\omega\,\vec{J}_c] + \vec{J}_s \tag{12}$$

where $\vec{J}_s$ is a source current with support in the half-space z>0. The quantity in brackets on the right side of (12) we denote as $\vec{D}_c$ and note from (10) and (11) that

$$\vec{D}_c = [(1 - h_1)\overset{1}{\varepsilon_2} + (h_1 - h_0)\overset{1}{\varepsilon_1} + h_0\varepsilon_0]\vec{E} \tag{13}$$

$$= \varepsilon^1 \vec{E}$$

where ε^1 is the general complex permittivity, and

$$\overset{1}{\varepsilon_2} = \varepsilon_2 + \frac{\sigma_2}{j\omega}, \quad \overset{1}{\varepsilon_1} = \varepsilon_1 + \frac{\sigma_1}{j\omega} . \tag{14}$$

We note that (13) may be inverted to give

$$\vec{E} = \left[(1 - h_1)\frac{1}{\overset{1}{\varepsilon_2}} + (h_1 - h_0)\frac{1}{\overset{1}{\varepsilon_1}} + h_0\frac{1}{\varepsilon_0}\right]\vec{D}_c$$

$$= [\varepsilon^1]^{-1}\vec{D}_c . \tag{15}$$

Now as usual we proceed by taking the curl of (3), i.e.

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu_0 \nabla \times \vec{H}$$

$$= -j\omega\mu_0 [j\omega\vec{D}_c + \vec{J}_s]$$

$$= \omega^2\mu_0\varepsilon_0[(1 - h_1)\frac{\overset{1}{\varepsilon_2}}{\varepsilon_0} + (h_1 - h_0)\frac{\overset{1}{\varepsilon^1}}{\varepsilon_0} + h_0]\vec{E} - j\omega\mu_0\vec{J}_s \tag{16}$$

or

$$\nabla \times \nabla \times \vec{E} - k^2[(1 - h_1)\eta_{02}^2 + (h_1 - h_0)\eta_{01}^2 + h_0]\vec{E} = -j\omega\mu_0\vec{J}_s \tag{17}$$

where

$$\eta_{02}^2 = \frac{\overset{1}{\varepsilon_2}}{\varepsilon_0} = \frac{\varepsilon_2}{\varepsilon_0} + \frac{\sigma_2}{j\omega\varepsilon_0} = \varepsilon_{r2} + \frac{\sigma_2}{j\omega\varepsilon_0} \tag{18}$$

$$\eta_{01}^2 = \frac{\overset{1}{\varepsilon_1}}{\varepsilon_0} = \frac{\varepsilon_1}{\varepsilon_0} + \frac{\sigma_1}{j\omega\varepsilon_0} = \varepsilon_{r1} + \frac{\sigma_1}{j\omega\varepsilon_0} \tag{19}$$

are the refractive indices squared and

$$k^2 = \omega^2\mu_0\varepsilon_0 \tag{20}$$

is the square of the free space wavenumber.

If we write

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla.\vec{E}) - \nabla^2\vec{E} \tag{21}$$

then (17) takes the form

$$\nabla^2\vec{E} + k^2[(1 - h_1)\eta_{02}^2 + (h_1 - h_0)\eta_{01}^2 + h_0]\vec{E} = j\omega\mu_0 \vec{J}_s + \nabla(\nabla.\vec{E}) \tag{22}$$

and we need to interpret $\nabla(\nabla.\vec{E})$

Refer back to (15) and note that this equation may be rewritten in the following form,

$$\vec{E} = \left[\frac{1}{\overset{1}{\varepsilon_2}} + \left(\frac{1}{\overset{1}{\varepsilon_1}} - \frac{1}{\overset{1}{\varepsilon_2}}\right) h_1 + \left(\frac{1}{\varepsilon_0} - \frac{1}{\overset{1}{\varepsilon_1}}\right) h_0\right] \vec{D}_c$$

$$= \left[\frac{1}{\overset{1}{\varepsilon_2}} + \left(\frac{\overset{1}{\varepsilon_2} - \overset{1}{\varepsilon_1}}{\overset{1}{\varepsilon_1}\ \overset{1}{\varepsilon_2}}\right) h_1 + \left(\frac{\overset{1}{\varepsilon_1} - \varepsilon_0}{\varepsilon_0\ \overset{1}{\varepsilon_1}}\right) h_0\right] \vec{D}_c \quad . \tag{23}$$

Then,

$$\nabla.\vec{E} = \frac{1}{\frac{1}{\varepsilon_2}} \nabla.\vec{D}_c + \left(\frac{\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}}{\frac{1}{\varepsilon_1}\frac{1}{\varepsilon_2}} \right) \nabla.(h_1\vec{D}_c)$$

$$+ \left(\frac{\frac{1}{\varepsilon_1} - \varepsilon_0}{\frac{1}{\varepsilon_0}\frac{1}{\varepsilon_1}} \right) \nabla.(h_0\vec{D}_c) \qquad (24)$$

or,

$$\nabla.\vec{E} = \frac{1}{\frac{1}{\varepsilon_2}} \nabla.\vec{D}_c + \left(\frac{\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}}{\frac{1}{\varepsilon_1}\frac{1}{\varepsilon_2}} \right) [h_1 \nabla.D_c + D_{cz}^{(1^+)}\delta(z+a_1)]$$

$$+ \left(\frac{\frac{1}{\varepsilon_1} - \varepsilon_0}{\frac{1}{\varepsilon_0}\frac{1}{\varepsilon_1}} \right) [h_0\nabla.\vec{D}_c + D_{cz}^{(0^+)}\delta(z)] \qquad (25)$$

where

$$D_{cz}^{(1^+)} = \lim_{z\to -a_1^+} D_{cz}, \qquad D_{cz}^{(0^+)} = \lim_{z\to 0^+} D_{cz}, \qquad (26)$$

and D_{cz} denotes the z-component of $\vec{D}_c$, i.e., the values of D_{cz} immediately above each interface. Now consider again (12) and recall, by assumption, the support of $\vec{J}_s$ is the half-space z>0. Then from (12),

$$\nabla.(\nabla\times\vec{H}) \equiv 0 = j\omega \nabla.\vec{D}_c + \nabla.\vec{J}_s \qquad (27)$$

i.e.,

$$\nabla.\vec{D}_c = h_1 \nabla.\vec{D}_c = h_0 \nabla.D_c = -\frac{1}{j\omega} \nabla.\vec{J}_s \qquad (28)$$

Therefore (25) may be rewritten as

$$\nabla.\vec{E} = -\frac{1}{j\omega\varepsilon_0}\nabla.\vec{J}_s + \left(\frac{{}^1\varepsilon_2 - {}^1\varepsilon_1}{{}^1\varepsilon_1\ {}^1\varepsilon_2}\right) D_{cz}(1^+)\ \delta(z+a_1)$$

$$+ \left(\frac{{}^1\varepsilon_1 - \varepsilon_0}{\varepsilon_0\ {}^1\varepsilon_1}\right) D_{cz}(0^+)\ \delta(z), \qquad (29)$$

and,

$$\nabla(\nabla.\vec{E}) = -\frac{1}{j\omega\varepsilon_0}\nabla(\nabla.\vec{J}_s) + \left(\frac{{}^1\varepsilon_2 - {}^1\varepsilon_1}{{}^1\varepsilon_1\ {}^1\varepsilon_2}\right) \nabla\left(D_{cz}(1^+)\ \delta(z+a_1)\right)$$

$$+ \left(\frac{{}^1\varepsilon_1 - \varepsilon_0}{\varepsilon_0\ {}^1\varepsilon_1}\right) \nabla(D_{cz}(0^+)\ \delta(z))$$

$$= -\frac{1}{j\omega\varepsilon_0}\nabla(\nabla.\vec{J}_s) + \left(\frac{{}^2\eta_{02} - {}^2\eta_{01}}{{}^2\eta_{02}}\right) \nabla(E_z(1^+)\ \delta(z+a_1))$$

$$+ \left(\frac{{}^2\eta_{01} - 1}{{}^2\eta_{01}}\right) \nabla(E_z(0^+)\delta(z)), \qquad (30)$$

using (18), (19) and (13).

Equation (30) is easily generalized to M (say) layers.

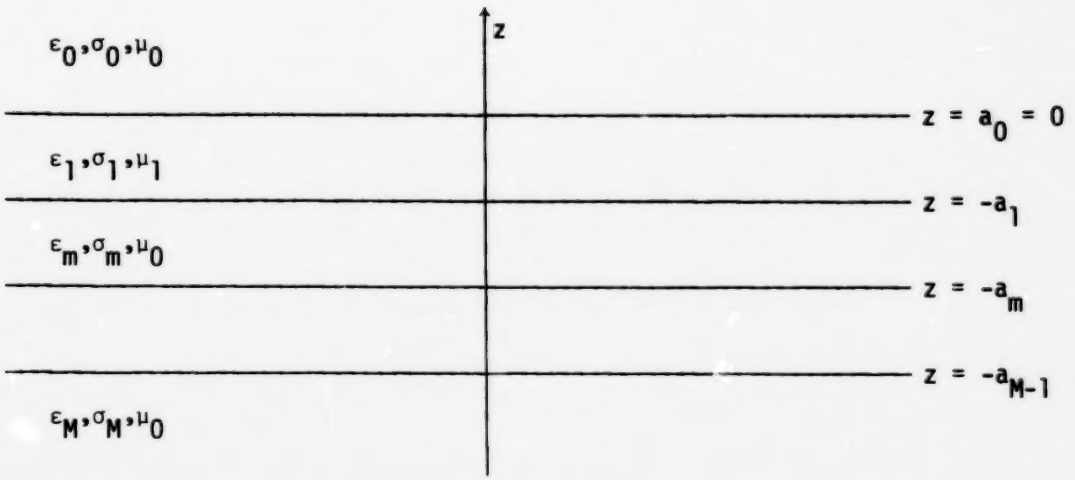

FIGURE 2 - M-LAYER MODEL

In this case, again assuming that the support $\vec{J}_s$ is contained in $z>0$, (30) becomes

$$\nabla(\nabla.\vec{E}) = -\frac{1}{j\omega\varepsilon_o}\nabla(\nabla.J_s) + \sum_{m=0}^{M-1}\left(\frac{\eta_{0(m+1)}^2 - \eta_{0m}^2}{\eta_{0(m+1)}^2}\right)\nabla[E_z^{(m+)}\delta(z+a_m)] \quad . \quad (31)$$

Note $\eta_{00} = 1$

It should be noted that (23) is not the only way to rewrite (15). From (15) we can also write

$$\vec{E} = \left[(1 - h_1)\left(\frac{\varepsilon_1^1 - \varepsilon_2^1}{\varepsilon_1^1\varepsilon_2^1}\right) + (1 - h_o)\left(\frac{\varepsilon_o^1 - \varepsilon_1^1}{\varepsilon_o^1\varepsilon_1^1}\right) + \frac{1}{\varepsilon_o}\right]\vec{D}_c. \quad (32)$$

Hence

$$\nabla.\vec{E} = \frac{1}{\varepsilon_o}\nabla.\vec{D}_c + \left(\frac{\varepsilon_1^1 - \varepsilon_2^1}{\varepsilon_1^1\varepsilon_2^1}\right)\nabla.[(1 - h_1)\vec{D}_c]$$

$$+ \left(\frac{\varepsilon_o^1 - \varepsilon_1^1}{\varepsilon_o^1\varepsilon_1^1}\right)\nabla.[(1 - h_o)\vec{D}_c]$$

$$= \frac{1}{\varepsilon_0} \nabla.\vec{D}_c + \left(\frac{\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}}{\frac{1}{\varepsilon_1}\ \frac{1}{\varepsilon_2}} \right) \{(1 - h_1)\ \nabla.\vec{D}_c - D_{cz}^{(1^-)} \delta(z+a_1)\}$$

$$+ \left(\frac{\varepsilon_0 - \varepsilon_1}{\frac{1}{\varepsilon_0}\ \frac{1}{\varepsilon_1}} \right) \{(1 - h_0)\ \nabla.\vec{D}_c - D_{cz}^{(0^-)}\ \delta(z)\}. \quad (33)$$

However, from (27)

$$(1 - h_1)\ \nabla.\vec{D}_c = (1 - h_0)\ \nabla.\vec{D}_c = 0$$

$$\nabla.\vec{D}_c = - \frac{1}{j\omega} \nabla.\vec{J}_s \ . \quad (34)$$

Therefore,

$$\nabla.\vec{E} = - \frac{1}{j\omega\varepsilon_0} \nabla.\vec{J}_s + \left(\frac{\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}}{\frac{1}{\varepsilon_1}\ \frac{1}{\varepsilon_2}} \right) D_{cz}^{(1^-)}\ \delta(z+a_1)$$

$$+ \left(\frac{\frac{1}{\varepsilon_1} - \varepsilon_0}{\frac{1}{\varepsilon_0}\ \frac{1}{\varepsilon_1}} \right) D_{cz}^{(0^-)}\ \delta(z) \quad (35)$$

In the above,

$$D_{cz}^{(1^-)} = \lim_{z \to -a_1^-} D_{cz} \ , \qquad D_{cz}^{(0^-)} = \lim_{z \to 0^-} D_{cz} \ ,$$

i.e., the values of D_{cz} immediately below each interface.

A solution using (29) implies, from uniqueness (35),

i.e.,

$$D_{cz}^{(1^-)} = D_{cz}^{(1^+)}, \quad D_{cz}^{(0^-)} = D_{cz}^{(0^+)} ,$$

or the continuity of D_{cz} at the interfaces. This agrees with classical theory.

The basic differential equation is then

$$\nabla^2\vec{E} + \gamma^2\vec{E} = -T_{SE}(\vec{J}_s) + \left(\frac{\eta_{02}^2 - \eta_{01}^2}{\eta_{02}^2} \right) \nabla(E_z^{(1^+)}\delta(z+a_1))$$

$$+ \left(\frac{\eta_{01}^2 - 1}{\eta_{01}^2} \right) \nabla \left(E_z^{(0^+)}\delta(z) \right) \qquad (36)$$

from (29) for the two layer model or, more generally, for the M layer model,

$$\nabla^2\vec{E} + \gamma^2\vec{E} = -T_{SE}(\vec{J}_s) + \sum_{m=0}^{M-1} \frac{\eta_{0(m+1)}^2 - \eta_{0m}^2}{\eta_{0(m+1)}^2} \nabla(E_z^{(m^+)}\delta(z+a_m)) \ . \qquad (37)$$

In (36) and (37)

$$T_{SE} = \frac{1}{J\omega\varepsilon_0} [\nabla(\nabla \cdot\ ) + k^2] \qquad (38)$$

is the "electrical source operator", ($k^2 = \omega^2\mu_0\varepsilon_0$).

In (36)

$$\gamma^2 = k^2 [\eta_{02}^2 (1 - h_1) + \eta_{01}^2(h_1 - h_0) + h_0]$$

$$= k^2 \sum_{m=0}^{2} \eta_{0m}^2 [h(z+a_m) - h(z+a_{m-1})], \qquad (39)$$

$a_{-1} = \infty$, $a_2 = -\infty$, $a_0 = 0$.

and in (37)

$$\gamma^2 = k^2 \sum_{m=0}^{M} [h(z+a_m) - h(z+a_{m-1})]\, \eta_{0m}^2 \quad . \tag{40}$$

(refer to Figure 2).

Equation (36), or (37), is then the basic partial differential equation that the electric field vector $\vec{E}$ must satisfy. A similar equation may be derived for the magnetic field $\vec{H}$. In the next section we describe a method of seeking solutions for (36) and (37).

2.4 The Field Decomposition

In the case of the two layer model we write $\vec{E}$ in the form

$$\vec{E} = (1 - h_1)\vec{E} + (h_1 - h_0)\vec{E} + h_0\vec{E}. \tag{41}$$

Then, for use in (36), we compute

$$\nabla^2\vec{E} = \nabla^2 [(1 - h)\vec{E}] + \nabla^2 [(h_1 - h_0)\vec{E}] + \nabla^2(h_0\vec{E}) \quad . \tag{42}$$

Now,

$$\nabla^2[(1 - h_1)\vec{E}] = (1 - h_1)\nabla^2\vec{E} - \vec{E}^{(1^-)}\delta'(z+a_1) - (\frac{\partial\vec{E}}{\partial z})^{(1^-)}\delta(z+a_1), \tag{43}$$

$$\nabla^2[(h_1 - h_0)E] = (h_1 - h_0)\nabla^2\vec{E} + \vec{E}^{(1^+)}\delta'(z+a_1)$$

$$+ (\frac{\partial\vec{E}}{\partial z})^{(1^+)}\delta(z+a_1) - \vec{E}^{(0^-)}\delta'(z) + (\frac{\partial\vec{E}}{\partial z})^{(0^-)}\delta(z), \tag{44}$$

and

$$\nabla^2(h_0\vec{E}) = h_0\nabla^2\vec{E} + \vec{E}^{(0^+)}\delta'(z) + (\frac{\partial\vec{E}}{\partial z})^{(0^+)}\delta(z) \tag{45}$$

where the superscripts have the same meaning as in (26) and (35).

Inserting (43), (44) and (45) into (36), it may be readily seen that a solution of (36) is given by $\vec{E}$ which satisfies,

$$h_o[\nabla^2\vec{E} + k^2\vec{E}] = -T_{SE}(\vec{J}_s) \tag{46}$$

$$(h_1 - h_o)[\nabla^2\vec{E} + \eta_{01}^2 k^2\vec{E}] = 0 \tag{47}$$

$$(1 - h_1)[\nabla^2\vec{E} + \eta_{02}^2 k^2\vec{E}] = 0 \tag{48}$$

$$[\vec{E}^{(1^+)} - \vec{E}^{(1^-)}]\delta'(z+a_1) + [(\frac{\partial\vec{E}}{\partial z})^{(1^+)} - (\frac{\partial\vec{E}}{\partial z})^{(1^-)}]\delta(z+a_1)$$

$$= \frac{\eta_{02}^2 - \eta_{01}^2}{\eta_{02}^2} \nabla[E_z^{(1^+)}\delta(z+a_1)] \tag{49}$$

$$[\vec{E}^{(0^+)} - \vec{E}^{(0^-)}]\delta'(z) + [(\frac{\partial\vec{E}}{\partial z})^{(0^+)} - (\frac{\partial\vec{E}}{\partial z})^{(0^-)}]\delta(z)$$

$$= \frac{\eta_{01}^2 - 1}{\eta_{01}^2} \nabla[E_z^{(0^+)}\delta(z)] \, . \tag{50}$$

It should be noted that (49) and (50) represent boundary conditions.

2.4.2 Reduction to Integral Equations

In this section the problem as represented by (46) - (50) will be reduced to one involving integral equations of the convolution type.

We first introduce the fundamental solutions.

$$K_o(r) = \frac{e^{-jkr}}{4\pi r} \tag{51}$$

$$K_1(r) = \frac{e^{-jk\eta_{01}r}}{4\pi r} \tag{52}$$

$$K_2(r) = \frac{e^{-jk\eta_{02}r}}{4\pi r}. \tag{53}$$

These functions satisfy

$$\nabla^2 K_o + k^2 K_o = -\delta(x)\,\delta(y)\,\delta(z) \tag{54}$$

$$\nabla^2 K_1 + k^2 \eta_{01}^2 K_1 = -\delta(x)\,\delta(y)\,\delta(z) \tag{55}$$

$$\nabla^2 K_2 + k^2 \eta_{02}^2 K_2 = -\delta(x)\,\delta(y)\,\delta(z) \tag{56}$$

and are subject to the Sommerfeld radiation condition at infinity.

First consider the identity

$$\nabla^2(h_o\vec{E})*K_o = h_o\vec{E}*\nabla^2 K_o \tag{57}$$

where * denotes three dimensional spatial convolution. The left-hand side is

$$= \{h_o\nabla^2\vec{E} + \vec{E}^{(0^+)}\delta'(z) + \left(\frac{\partial E}{\partial z}\right)^{(0^+)}\delta(z)\}*K_o \tag{58}$$

while the right-hand side is

$$= h_o\vec{E}*\{-\delta(x)\,\delta(y)\,\delta(z) - k^2 K_o\}$$

$$= -h_o\vec{E} - (h_o k^2\vec{E})*K_o \tag{59}$$

Hence, equating (58) and (59), we obtain

$$h_o\vec{E} = -h_o[\nabla^2\vec{E} + k^2\vec{E}]*K_o - [\vec{E}^{(0^+)}\delta'(z) + \left(\frac{\partial\vec{E}}{\partial z}\right)^{(0^+)}\delta(z)]*K_o\,. \tag{60}$$

Therefore, if (46) is satisfied

$$h_o\vec{E} = T_{SE}(\vec{J}_s)*K_o - [\vec{E}^{(0^+)}\delta'(z) + (\frac{\partial\vec{E}}{\partial z})^{(0^+)}\delta(z)]*K_o \quad . \tag{61}$$

In the same way we obtain

$$(h_1 - h_o)\vec{E} = [\vec{E}^{(0^-)}\delta'(z) + (\frac{\partial\vec{E}}{\partial z})^{(0^-)}\delta(z)]*K_1$$

$$-[\vec{E}^{(1^+)}\delta'(z+a_1) + (\frac{\partial\vec{E}}{\partial z})^{(1^+)}\delta(z+a_1)]*K_1 \tag{62}$$

and

$$(1 - h_1)\vec{E} = [\vec{E}^{(1^-)}\delta'(z+a_1) + (\frac{\partial\vec{E}}{\partial z})^{(1^-)}\delta(z+a_1)]*K_2 \quad . \tag{63}$$

The problem now may be viewed as one of using (61) - (63) and the boundary condition equations (49) and (50), to determine the unknown function $\vec{E}^{(0^+)}$, $(\frac{\partial\vec{E}}{\partial z})^{(0^+)}$, etc. If these are determined, (61) - (63) may be used to find the fields in all regions.

2.4.3 Generation and Solution of the Integral Equations

We first use the boundary equations to eliminate the quantities with negative superscripts in (61) - (63). From (49) and (50),

$$\vec{E}^{(1^-)}\delta'(z+a_1) + (\frac{\partial\vec{E}}{\partial z})^{(1^-)}\delta(z+a_1)$$

$$= \vec{E}^{(1^+)}\delta'(z+a_1) + (\frac{\partial\vec{E}}{\partial z})^{(1^+)}\delta(z+a_1) - N_{21}\nabla[E_z{}^{(1^+)}\delta(z+a_1)] \tag{64}$$

$$\vec{E}^{(0^-)}\delta'(z) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(0^-)}\delta(z)$$

$$= \vec{E}^{(0^+)}\delta'(z) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(0^+)}\delta(z) - N_{10}\nabla[E_z{}^{(0^+)}\delta(z)] \tag{65}$$

where

$$N_{21} = \frac{\eta_{02}^2 - \eta_{01}^2}{\eta_{02}^2} \tag{66}$$

$$N_{10} = \frac{\eta_{01}^2 - 1}{\eta_{01}^2} \tag{67}$$

Substituting (64) and (65) into (62) and (63), (61) - (63) may be rewritten as,

$$h_o\vec{E} = \vec{E}_s - [\vec{E}^{(0^+)}\delta'(z) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(0^+)}\delta(z)]*K_o \tag{68}$$

$$(h_1 - h_o)\vec{E} = [\vec{E}^{(0^+)}\delta'(z) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(0^+)}\delta(z)]*K_1 - N_{10}\nabla[E_z{}^{(0^+)}\delta(z)]*K_1$$

$$- [\vec{E}^{(1^+)}\delta'(z+a_1) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(1^+)}\delta(z+a_1)]*K_1 \tag{69}$$

$$(1 - h_1)\vec{E} = [\vec{E}^{(1^+)}\delta'(z+a_1) + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(1^+)}\delta(z+a_1)]*K_2$$

$$- N_{21}\nabla[E_z{}^{(1^+)}\delta(z+a_1)]*K_2 \quad . \tag{70}$$

There are four unknown quantities in (68) - (70). However, four equations may be obtained by choosing appropriate planes above and below each interface which make the left hand sides of (68) - (70)

equal to zero. Thus, one equation is obtained from (68), two from (69) and one from (70). Before proceeding, however, it is convenient, because of the "convolution" nature of the equations, to take two-dimensional (x and y) spatial Fourier transforms. The transforms of the fundamental solutions (51) -(53) are,

$$\underline{K}_0(k_x,k_y) = \frac{e^{-|z|\mu_0}}{2\mu_0} \tag{71}$$

$$\underline{K}_1(k_x,k_y) = \frac{e^{-|z|\mu_1}}{2\mu_0} \tag{72}$$

$$\underline{K}_2(k_x,k_y) = \frac{e^{-|z|\mu_2}}{2\mu_0} \tag{73}$$

where

$$\mu_0 = \sqrt{\lambda^2 - k^2}$$

$$\mu_1 = \sqrt{\lambda^2 - (\eta_{01}k)^2}$$

$$\mu_2 = \sqrt{\lambda^2 - (\eta_{02}k)^2}$$

$$\lambda^2 = k_x^2 + k_y^2$$

and k_x, k_y are the transformed variables, i.e.,

$$\underline{f}(k_x, k_y) = \int_x \int_y f(x,y)\, e^{-j(k_x x + k_y y)} dxdy.$$

Thus, (68) - (70) become

$$h_0 \vec{E} = \underline{\vec{E}_s} - [\underline{\vec{E}}^{(0^+)} \frac{\partial}{\partial z} + (\underline{\frac{\partial \vec{E}}{\partial z}})^{(0^+)}] (\frac{e^{-|z|\mu_0}}{2\mu_0}) \tag{74}$$

$$(h_1 - h_o)\vec{E} = [(\vec{E}^{(0^+)} - N_{10}\,\vec{E}_z^{(0^+)}\vec{a}_z)\,\frac{\partial}{\partial z} + \{(\frac{\partial \vec{E}}{\partial z})^{(0^+)} - jN_{10}(k_x\vec{a}_x + k_y\vec{a}_y)$$

$$\cdot E_z^{(0^+)}\}]\,(\frac{e^{-|z|\mu_1}}{2\mu_1})$$

$$-[\vec{E}^{(1^+)}\,\frac{\partial}{\partial z} + (\frac{\partial \vec{E}}{\partial z})^{(1^+)}]\,(\frac{e^{-|z+a_1|\mu_1}}{2\mu_1}) \qquad (75)$$

$$(1 - h_1)\vec{E} = [(\vec{E}^{(1^+)} - N_{21}E_z^{(1^+)}\,\vec{a}_z)\,\frac{\partial}{\partial z} +$$

$$\{\frac{\partial \vec{E}^{(1^+)}}{\partial z} - jN_{21}(k_x\vec{a}_x + k_y\vec{a}_y)E_z^{(1^+)}\}](\frac{e^{-|z+a_1|\mu_2}}{2\mu_2})\ . \qquad (76)$$

We now choose a plane (say $z = z^- < 0$) such that the left side of (74) is zero, i.e.

$$\vec{E}_s z^- = \vec{E}^{(0^+)}\ (\frac{e^{-|z^-|\mu_o}}{2}) + (\frac{\partial \vec{E}}{\partial z})^{(0^+)}\ (\frac{e^{-|z^-|\mu_o}}{2\mu_o})$$

or

$$2\mu_o e^{|z^-|\mu_o}\,\vec{E}_s z^- = \mu_o\vec{E}^{(0^+)} + (\frac{\partial \vec{E}}{\partial z})^{(0^+)}\ . \qquad (77)$$

Then in (75), choose a plane (say $z = z^+ > 0$) such that the left side is zero, thus

$$0 = -(\vec{E}^{(0^+)} - N_{10}E_z^{(0^+)}\vec{a}_z)\,(\frac{e^{-|z^+|\mu_1}}{2})$$

$$+ \{(\frac{\partial \vec{E}}{\partial z})^{(0^+)} - jN_{10}(k_x\vec{a}_x + k_y\vec{a}_y)E_z^{(0^+)}\}\,(\frac{e^{-|z^+|\mu_1}}{2\mu_1})$$

$$+ \vec{E}^{(1^+)}\,(\frac{e^{-|z^+ + a_1|\mu_1}}{2}) - (\frac{\partial \vec{E}}{\partial z})^{(1^+)}(\frac{e^{-|z^+ + a_1|\mu_1}}{2\mu_1})$$

or

$$0 = -\mu_1 \underline{\vec{E}}^{(0^+)} + (\mu_1 \vec{a}_z - jk_x \vec{a}_x - jk_y \vec{a}_y) N_{10} \underline{E_z}^{(0^+)} + (\underline{\frac{\partial \vec{E}}{\partial z}})^{(0^+)}$$

$$+ \mu_1 e^{-a_1 \mu_1} \underline{\vec{E}}^{(1^+)} - (\underline{\frac{\partial \vec{E}}{\partial z}})^{(1^+)} e^{-a_1 \mu_1}. \qquad (78)$$

Also in (75), choose a plane ($z < -a_1$) such that the left side is zero, i.e.

$$0 = (\underline{\vec{E}}^{(0^+)} - N_{10} \underline{E_z}^{(0^+)} \vec{a}_z) \left(\frac{e^{-|z|\mu_1}}{2}\right) + [(\underline{\frac{\partial \vec{E}}{\partial z}})^{(0^+)} - jN_{10}(k_x \vec{a}_x + k_y \vec{a}_y)$$

$$\cdot \underline{E_z}^{(0^+)}] \left(\frac{e^{-|z|\mu_1}}{2\mu_1}\right) - \underline{\vec{E}}^{(1^+)} \left(\frac{e^{-|z+a_1|\mu_1}}{2}\right) - (\underline{\frac{\partial \vec{E}}{\partial z}})^{(1^+)} \left(\frac{e^{-|z+a_1|\mu_1}}{2\mu_1}\right)$$

$$0 = \mu_1 \underline{\vec{E}}^{(0^+)} + (-\mu_1 \vec{a}_z - jk_x \vec{a}_x - jk_y \vec{a}_y) N_{10} \underline{E_z}^{(0^+)} + (\underline{\frac{\partial \vec{E}}{\partial z}})^{(0^+)}$$

$$- \mu_1 \underline{\vec{E}}^{(1^+)} e^{a_1 \mu_1} - (\underline{\frac{\partial \vec{E}}{\partial z}})^{(1^+)} e^{a_1 \mu_1}. \qquad (79)$$

Finally, in (76), we choose a plane ($z > -a_1$) such that the left side is zero, i.e.,

$$0 = -(\underline{\vec{E}}^{(1^+)} - N_{21} \underline{E_z}^{(1^+)} \vec{a}_z \left(\frac{e^{-|z+a_1|\mu_2}}{2}\right)$$

$$+ [(\underline{\frac{\partial \vec{E}}{\partial z}})^{(1^+)} - jN_{21} (k_x \vec{a}_x + k_y \vec{a}_y) \underline{E_z}^{(1^+)}] \left(\frac{e^{-|z+a_1|\mu_2}}{2\mu_2}\right)$$

or

$$0 = -\mu_2 \underline{\vec{E}}^{(1^+)} + (\mu_2 \vec{a}_z - jk_x \vec{a}_x - jk_y \vec{a}_y)\underline{E}_z^{(1^+)} N_{21} + \left(\frac{\partial \vec{E}}{\partial z}\right)^{(1^+)} . \qquad (80)$$

Equations (77), (78), (79) and (80) may now be solved simultaneously for the unknown terms $\underline{\vec{E}}^{(0^+)}$, $\frac{\partial \vec{E}}{\partial z}^{(0^+)}$, $\underline{\vec{E}}^{(1^+)}$ and $\frac{\partial \vec{E}}{\partial z}^{(1^+)}$. To this end, we will introduce a matrix notation, thus (80) may be rewritten as

$$\underline{D_z E}^{(1^+)} = (U_2 B_2 + D_2)\underline{E}^{(1^+)} \qquad (81)$$

where

$$\underline{D_z E}^{(1^+)} = \begin{bmatrix} \left(\frac{\partial E_x}{\partial z}\right)^{(1^+)} \\ \left(\frac{\partial E_y}{\partial z}\right)^{(1^+)} \\ \left(\frac{\partial E_z}{\partial z}\right)^{(1^+)} \end{bmatrix} \qquad \underline{E}^{(1^+)} = \begin{bmatrix} \underline{E_x}^{(1^+)} \\ \underline{E_y}^{(1^+)} \\ \underline{E_z}^{(1^+)} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(\frac{\eta_{01}}{\eta_{02}}\right)^2 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0 & 0 & jN_{21} k_x \\ 0 & 0 & jN_{21} k_y \\ 0 & 0 & 0 \end{bmatrix}$$

Also, a simultaneous solution of (79) and (80) yields,

$$\underline{E}^{(1^+)} = [c_1 I + s_1(U_2B_2 + D_2)]^{-1}B_1\underline{E}^{(0^+)} \tag{82}$$

$$\underline{D_z E}^{(0^+)} = [\mu_1 t_1 I + (U_2B_2 + D_2)][I + \frac{t_1}{\mu_1}(U_2B_2 + D_2)]^{-1}B_1\underline{E}^{(0^+)} + D_1\underline{E}^{(0^+)}$$

$$= [U_1B_1 + D_1]\underline{E}^{(0^+)} \tag{83}$$

where

$$\underline{E}^{(0^+)} = \begin{bmatrix} \underline{E}_x^{(0^+)} \\ \underline{E}_y^{(0^+)} \\ \underline{E}_z^{(0^+)} \end{bmatrix} \qquad \underline{D_z E}^{(0^+)} = \begin{bmatrix} (\frac{\partial E_x}{\partial z})^{(0^+)} \\ (\frac{\partial E_y}{\partial z})^{(0^+)} \\ (\frac{\partial E_z}{\partial z})^{(0^+)} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\frac{\eta_{00}}{\eta_{01}})^2 \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 0 & jN_{10}k_x \\ 0 & 0 & jN_{10}k_y \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 = \cosh(a_1\mu_1)$

$s_1 = \sinh(a_1\mu_1)$

$t_1 = \tanh(a_1\mu_1)$

and I is the 3 x 3 identity matrix. The advantages of this notation will be apparent in the solution for the M layer structure.

Refer to (77), which we rewrite as

$$2\mu_o\underline{E_{os}}^{z^-} = \mu_o\underline{E}^{(0^+)} + \underline{D_z E}^{(0^+)} \tag{84}$$

where

$$\underline{E}_{os} = e^{|z^-|\mu_o} \begin{bmatrix} \underline{E}_{sx}{}^{z^-} \\ \underline{E}_{sy}{}^{z^-} \\ \underline{E}_{sz}{}^{z^-} \end{bmatrix} .$$

Using (83), we have

$$2\mu_o \underline{E}_{os} = \mu_o \underline{E}^{(0^+)} + [U_1 B_1 + D_1]\underline{E}^{(0^+)}$$

or,

$$\underline{E}^{(0^+)} = [\mu_o I + (U_1 B_1 + D_1)]^{-1} (2\mu_o) \underline{E}_{os} \tag{85}$$

This completes the solution for the surface field quantities for the 2-layer case.

2.5 The Electric Field $\vec{E}$ in the Half-Space Above the Layers

Equation (83) and (85) may now be used in conjunction with (74) to find an expression for the electric field $\vec{E}$ above the layered media.

Consider (74) and choose a plane ($z^+ > 0$), then

$$\underline{E}^{(z^+)} = \underline{E}_s{}^{(z^+)} + \frac{e^{-z^+\mu_o}}{2\mu_o} [\mu_o \underline{E}^{(0^+)} - \underline{D}_z \underline{E}^{(0^+)} \tag{86}$$

Hence, from (83),

$$\underline{E}^{(z+)} = \underline{E}_s{}^{(z+)} + \frac{e^{-z^+\mu_o}}{2\mu_o} [\mu_o I - (U_1 B_1 + D_1)]\underline{E}^{(0^+)}]$$

and from (85)

$$\underline{E}(z^+) = \underline{E_s}(z^+) + \frac{e^{-z^+\mu_o}}{2\mu_o}[\mu_o I - (U_1B_1 + D_1)] \cdot$$

$$\cdot [\mu_o I + (U_1B_1 + D_1)]^{-1}(2\mu_o)\underline{E_{os}}$$

$$= \underline{E_s}(z^+) + e^{-z^+\mu_o}[\mu_o I - (U_1B_1 + D_1)][\mu_o I + (U_1B_1 + D_1)]^{-1}\underline{E_{os}}. \quad (87)$$

Equation (87) is then the formal solution in the transformed variables k_x and k_y. Taking the inverse two-dimensional spatial Fourier transform gives

$$E(z^+) = E_s(z^+) + \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}[\mu_o I - (U_1B_1 + D_1)][\mu_o I$$

$$+ (U_1B_1 + D_1)]^{-1}\,\underline{E_{os}}\, e^{-z+\mu_o+jk_x x+jk_y y}\,dk_x dk_y \quad . \quad (88)$$

2.6 The Electric Field $\vec{E}$ in the Half-Space Above the Layers, for the M Layer Case

In this case also, the general solution is given by (87) and (88), where the various quantities are given by (83). However, now the matrices U_m (m = M, 1) are calculated recursively as

$$U_m = [\mu_m t_m I + (U_{m+1}B_{m+1} + D_{m+1})][I + \frac{t_m}{\mu_m}(U_{m+1}B_{m+1} + D_{m+1})]^{-1} \quad (89)$$

where

$$\mu_m = \sqrt{k_x^2 + k_y^2 - n_{Om}^2 k^2} \quad = \sqrt{\lambda^2 - n_{Om}^2 k^2}$$

$$t_m = \tanh[(a_m - a_{m-1})\mu_m]$$

$$B_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(\frac{\eta_{0(m-1)}}{\eta_{0m}}\right)^2 \end{bmatrix} \qquad D_m = \begin{bmatrix} 0 & 0 & jN_{m(m-1)}k_x \\ 0 & 0 & jN_{m(m-1)}k_y \\ 0 & 0 & 0 \end{bmatrix}$$

$$\eta_{0m}^2 = \varepsilon_{rm} - j\frac{\sigma_m}{\omega\varepsilon_o}$$

$$N_{m(m-1)} = \frac{\eta_{0m}^2 - \eta_{0(m-1)}^2}{\eta_{0m}^2} \quad .$$

It should be noted that

$$U_M = \begin{bmatrix} \mu_M & 0 & 0 \\ 0 & \mu_M & 0 \\ 0 & 0 & \mu_M \end{bmatrix} \quad . \tag{90}$$

2.7 The Vertical Electric Field E_z

If the only required information is the vertical component of the electric field E_z, the above calculation may be simplified. To see this, let us rewrite (77), (78), (79) and (80) for the z - component only, i.e.,

$$2\mu_o e^{|z^-|\mu_o} \underline{E_{sz}} z^- = \mu_o \underline{E_z}^{(0^+)} + \left(\frac{\partial \vec{E}}{\partial z}\right)_z^{(0^+)} \tag{91}$$

$$0 = -\mu_1(1 - N_{10})\underline{E_z}^{(0^+)} + \left(\frac{\partial \vec{E}}{\partial z}\right)_z^{(0^+)}$$

$$+ \mu_1 e^{-a_1\mu_1}\underline{E_z}^{(1^+)} - \left(\frac{\partial \vec{E}}{\partial z}\right)_z^{(1^+)} e^{-a_1\mu_1} \tag{92}$$

$$0 = \mu_1(1 - N_{10})\underline{E_z}^{(0^+)} + (\underline{\frac{\partial \vec{E}}{\partial z}})_z^{(0^+)} - \mu_1 e^{a_1\mu_1}\underline{E_z}^{(1^+)} - (\underline{\frac{\partial \vec{E}}{\partial z}})_z^{(1^+)} e^{a_1\mu_1} \quad (93)$$

$$(\underline{\frac{\partial \vec{E}}{\partial z}})_z^{(1^+)} = \mu_2(1 - N_{21})\underline{E_z}^{(1^+)} = U_{2z}(1 - N_{21})\underline{E_z}^{(1^+)} \quad (94)$$

Using (94) in (92) and (93), and solving these simultaneously for $\underline{\frac{\partial \vec{E}}{\partial z}}^{(0^+)}$ and $\underline{E_z}^{(1^+)}$ gives

$$\underline{E_z}^{(1^+)} = \frac{(1 - N_{10})}{c_1 + \frac{\mu_2}{\mu_1}(1 - N_{21})s_1}\ \underline{E_z}^{(0^+)} \quad (95)$$

$$(\underline{\frac{\partial \vec{E}}{\partial z}})_z^{(0^+)} = \frac{[\mu_1 t_1 + \mu_2(1 - N_{21})]}{1 + t_1 \frac{\mu_2}{\mu_1}(1 - N_{21})}(1 - N_{10})\ \underline{E_z}^{(0^+)}$$

$$= \frac{[\mu_1 t_1 + U_{2z}(1 - N_{21})]}{1 + t_1 \frac{U_{2z}}{\mu_1}(1 - N_{21})}(1 - N_{10})\ \underline{E_z}^{(0^+)}$$

$$= U_{1z}(1 - N_{10})\underline{E_z}(0^+) \quad (96)$$

The various terms are defined in (83) and (89). Equations (87) and (88) are then, for the z-component only,

$$\underline{E_z}^{(z^+)} = \underline{E_{sz}}^{(z^+)} + e^{-z^+\mu_o}\left[\frac{\mu_o - U_{1z}(1 - N_{10})}{\mu_o + U_{1z}(1 - N_{10})}\right]\underline{E_{osz}} \quad (97)$$

$$E_z^{(z^+)} = E_{sz}^{(z^+)} + \frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}\left[\frac{\mu_o - U_{1z}(1 - N_{10})}{\mu_o + U_{1z}(1 - N_{10})}\right]\underline{E_{osz}}$$

$$e^{-z^+\mu_o + jk_x x + jk_y y}\, dk_x dk_y \quad (98)$$

where $\underline{E}_{oxz}$ is as in (84).

Again, the general solution for the M layer case is given by (97) and (98). In this case, we recursively compute

$$U_{mz} = \frac{[\mu_m t_m + U_{(m+1)z}(1 - N_{(m+1)m})]}{1 + t_m \frac{U_{(m+1)z}}{\mu_m}(1 - N_{(m+1)m})} \tag{99}$$

where the various terms are given in (89).

2.8 Recursive Calculation of Equation (89) and the Resulting Fields

In this section we will give a method for calculating the quantities U_m of equation (89), which avoids the matrix inversion at each step. Further, the expression for the electrical field, i.e. (87) and (88) will also be given.

We shall assume an M layer structure and consider U_{M-1} in (89). Now,

$$U_M = \begin{bmatrix} \mu_M & 0 & 0 \\ 0 & \mu_M & 0 \\ 0 & 0 & \mu_M \end{bmatrix} \qquad B_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{n^2_{O(M-1)}}{n^2_{OM}} \end{bmatrix}$$

$$D_M = \begin{bmatrix} 0 & 0 & jk_x N_{M(M-1)} \\ 0 & 0 & jk_x N_{M(M-1)} \\ 0 & 0 & 0 \end{bmatrix} . \tag{100}$$

Therefore,

$$I + \frac{t_{M-1}}{\mu_{M-1}} (U_M B_M + D_M) =$$

$$\begin{bmatrix} 1 + t_{M-1} \frac{\mu_M}{\mu_{M-1}} & 0 & jk_x t_{M-1} \frac{N_{M(M-1)}}{\mu_{M-1}} \\ 0 & 1 + t_{M-1} \frac{\mu_M}{\mu_{M-1}} & jk_y t_{M-1} \frac{N_{M(M-1)}}{\mu_{M-1}} \\ 0 & 0 & 1 + t_{M-1} \frac{\mu_M}{\mu_{M-1}} \frac{n^2_{0(M-1)}}{n^2_{0M}} \end{bmatrix} . \quad (101)$$

The inverse of this matrix is,

$$[I + \frac{t_{M-1}}{\mu_{M-1}} (U_M B_M + D_M)]^{-1} =$$

$$\begin{bmatrix} \frac{1}{1 + t_{M-1} \frac{\mu_M}{\mu_{M-1}}} & 0 & \frac{-jk_x t_{M-1} \frac{N_{M(M-1)}}{\mu_{M-1}}}{(1+t_{M-1} \frac{\mu_M}{\mu_{M-1}})(1+t_{M-1} \frac{n^2_{0(M-1)}}{n^2_{0M}} \frac{\mu_M}{\mu_{M-1}})} \\ 0 & \frac{1}{1 + t_{M-1} \frac{\mu_M}{\mu_{M-1}}} & \frac{-jk_y t_{M-1} \frac{N_{M(M-1)}}{\mu_{M-1}}}{(1+t_{M-1} \frac{\mu_M}{\mu_{M-1}})(1+t_{M-1} \frac{n^2_{0(M-1)}}{\eta^2_{0M}} \frac{\mu_M}{\mu_{M-1}})} \\ 0 & 0 & \frac{1}{1+t_{M-1} \frac{n^2_{0(M-1)}}{n^2_{0M}} \frac{\mu_M}{\mu_{M-1}}} \end{bmatrix} . \quad (102)$$

Further, referring to (89)

$$[\mu_{M-1}t_{M-1}\, I + (U_M B_M + D_M)] =$$

$$\begin{bmatrix} \mu_{M-1}t_{M-1} + \mu_M & 0 & jk_x N_{M(M-1)} \\ 0 & \mu_{M-1}t_{M-1} + \mu_M & jk_y N_{M(M-1)} \\ 0 & 0 & \mu_{M-1}t_{M-1} + \mu_M \dfrac{n^2_{0(M-1)}}{n^2_{0M}} \end{bmatrix} . \quad (103)$$

Taking then the indicated product in (89) gives

$$U_{M-1} = \begin{bmatrix} U_{(M-1)h} & 0 & jk_x \nu_{M-1} \\ 0 & U_{(M-1)h} & jk_y \nu_{M-1} \\ 0 & 0 & U_{(M-1)z} \end{bmatrix} , \quad (104)$$

where

$$U_{(M-1)h} = \frac{\mu_{M-1}t_{M-1} + \mu_M}{1 + t_{M-1}\dfrac{\mu_M}{\mu_{M-1}}}$$

$$U_{(M-1)z} = \frac{\mu_{M-1}t_{M-1} + \mu_M \dfrac{n^2_{0(M-1)}}{n^2_{0M}}}{1 + t_{M-1}\dfrac{n^2_{0(M-1)}}{n^2_{0M}}\dfrac{\mu_M}{\mu_{M-1}}}$$

$$\nu_{M-1} = \frac{(1-t^2_{M-1})\, N_{M(M-1)}}{(1+t_{M-1}\dfrac{n^2_{0(M-1)}}{n^2_{0M}}\dfrac{\mu_M}{\mu_{M-1}})(1+t_{M-1}\dfrac{\mu_M}{\mu_{M-1}})} .$$

Then, repeating the above calcuation as indicated in (89) gives

$$U_{M-2} = \begin{bmatrix} U_{(M-2)h} & 0 & jk_x\nu_{M-2} \\ 0 & U_{(M-2)h} & jk_y\nu_{M-2} \\ 0 & 0 & U_{(M-2)z} \end{bmatrix} \tag{105}$$

where

$$U_{(M-2)h} = \frac{\mu_{M-2}t_{M-2} + U_{(M-1)h}}{1+t_{M-2}\dfrac{U_{(M-1)h}}{\mu_{M-2}}}$$

$$U_{(M-2)z} = \frac{\mu_{M-2}t_{M-2} + \dfrac{n^2_{0(M-2)}}{n^2_{0(M-1)}}U_{(M-1)z}}{1+t_{M-2}\dfrac{n^2_{0(M-2)}}{n^2_{0}(m-1)}\dfrac{U_{(M-1)z}}{\mu_{M-2}}}$$

$$\nu_{M-2} = \frac{(1 - t^2_{M-2})(\dfrac{n^2_{0(M-2)}}{n^2_{0(M-1)}}\nu_{M-1} + N_{(M-1)(M-2)})}{(1+t_{M-2}\dfrac{U_{(M-1)h}}{\mu_{M-2}})(1+t_{M-2}\dfrac{n^2_{0(M-2)}}{n^2_{0(M-1)}}\dfrac{U_{(M-1)z}}{\mu_{M-2}})} .$$

We therefore have the following formula for computing the matrices U_m in (89).

$$U_m = \begin{bmatrix} U_{mh} & 0 & jk_x\nu_m \\ 0 & U_{mh} & jk_y\nu_m \\ 0 & 0 & U_{mz} \end{bmatrix} \tag{106}$$

where

$$U_{mh} = \frac{\mu_m t_m + U_{(m+1)h}}{1 + t_m\dfrac{U_{(m+1)h}}{\mu_m}} \tag{107}$$

$$U_{mz} = \frac{\mu_m t_m + \dfrac{n_{0m}^2}{n_{0(m+1)}^2} U_{(m+1)z}}{1+t_m \dfrac{n_{0m}^2}{n_{0(m+1)}^2} \dfrac{U_{(m+1)z}}{\mu_m}} \tag{108}$$

$$\nu_m = \frac{(1-t_m^2)\left(\dfrac{n^2_{0m}}{n^2_{0(m+1)}} \nu_{m+1} + N_{(m+1)m}\right)}{\left(1+t_m \dfrac{U_{(m+1)h}}{\mu_m}\right)\left(1+t_m \dfrac{n_{0m}^2}{n_{0(m+1)}^2} \dfrac{U_{(m+1)z}}{\mu_m}\right)} \tag{109}$$

Refer now to (87), the equation for the electric field. From (106)

$$U_1 = \begin{bmatrix} U_{1h} & 0 & jk_x\nu_1 \\ 0 & U_{1h} & jk_y\nu_1 \\ 0 & 0 & U_{1z} \end{bmatrix}, \tag{110}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{n_{01}^2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & jk_x N_{10} \\ 0 & 0 & jk_y N_{10} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$[\mu_0 I + (U_1 B_2 + D_1)] =$$

$$\begin{bmatrix} \mu_0 + U_{1h} & 0 & jk_x\left(\dfrac{1}{n_{01}^2}\nu_1 + N_{10}\right) \\ 0 & \mu_0 + U_{1h} & jk_y\left(\dfrac{1}{n_{01}^2}\nu_1 + N_{10}\right) \\ 0 & 0 & \mu_0 + \dfrac{1}{n_{01}^2} U_{1z} \end{bmatrix} \tag{111}$$

The inverse is

$$[\mu_0 I + (U_1 B_1 + D_1)]^{-1} =$$

$$\begin{bmatrix} \dfrac{1}{\mu_0 + U_{1h}} & 0 & \dfrac{-jk_x(\dfrac{1}{n_{01}^2}\nu_1 + N_{10})}{(\mu_0 + U_{1h})(\mu_0 + \dfrac{1}{n_{01}^2}U_{1z})} \\ 0 & \dfrac{1}{\mu_0 + U_{1h}} & \dfrac{-jk_y(\dfrac{1}{n_{01}^2}\nu_1 + N_{10})}{(\mu_0 + U_{1h})(\mu_0 + \dfrac{1}{n_{01}^2}U_{1z})} \\ 0 & 0 & \dfrac{1}{\mu_0 + \dfrac{1}{n_{01}^2}U_{1z}} \end{bmatrix} . \qquad (112)$$

Also

$$[\mu_0 I - (U_1 B_1 + D_1)] =$$

$$\begin{bmatrix} \mu_0 - U_{1h} & 0 & jk_x(\dfrac{1}{n_{01}^2}\nu_1 + N_{10}) \\ 0 & \mu_0 - U_{1h} & -jk_y(\dfrac{1}{n_{01}^2}\nu_1 + N_{10}) \\ 0 & 0 & \mu_0 - \dfrac{1}{n_{01}^2}U_{1z} \end{bmatrix} . \qquad (113)$$

Carrying out the required product in (87) yields

$$[\mu_0 I - (U_1 B_1 + D_1)][\mu_0 I + (U_1 B_1 + D_1)]^{-1} =$$

$$\begin{bmatrix} \dfrac{\mu_0 - U_{1h}}{U_o + U_{1h}} & 0 & -jk_x \dfrac{2\mu_0(\frac{1}{n_{01}^2}\nu_1 + N_{10})}{(\mu_0 + U_{1h})(\mu_0 + \frac{1}{n_{01}^2}U_{1z})} \\ 0 & \dfrac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}} & -jk_y \dfrac{2\mu_0(\frac{1}{n_{01}^2}\nu_1 + N_{10})}{(\mu_0 + U_{1h})(\mu_0 + \frac{1}{n_{01}^2}U_{1z})} \\ 0 & 0 & \dfrac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}} \end{bmatrix}. \quad (114)$$

Therefore, using (114). we may write for the field components in (87),

$$\underline{E}_x(z^+) = \underline{E}_{sx}(z^+) + e^{-z^+\mu_o}\left[\frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\underline{E}_{osx} - jk_x \frac{2\mu_0 \frac{1}{n_{01}^2}\nu_1 + N_{10}}{(\mu_0 + U_{1h})(\mu_0 + \frac{1}{n^2_{01}}U_{1z})}\underline{E}_{osz}\right]$$

$$\underline{E}_y(z^+) = \underline{E}_{sy}(z^+) + e^{-z^+\mu_0}\left[\frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\underline{E}_{osy} - jk_y \frac{2\mu_o(\frac{1}{n_{01}^2}\nu_1 + N_{10})}{(\mu_0 + U_{1h})(\mu_0 + \frac{1}{n_{01}^2}U_{1z})}\underline{E}_{osz}\right] \quad (115)$$

and

$$\underline{E}_z(z^+) = \underline{E}_{sz}(z^+) + e^{-z^+\mu_0} \frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\underline{E}_{osz}$$

where the vector $\underline{E}_{os}$ is defined in (84).

Finally, to close this section we shall prove that

$$\mu_m^2 - U_{mh}U_{mz} = \lambda^2 \nu_m \tag{116}$$

This identity will be found useful in the sequel.

We shall prove (116) by induction. First, let m = M -1. Then from (104),

$$\mu_{M-1}^2 - U_{(M-1)h}U_{(M-1)z} = \mu_{M-1}^2 - \left[\frac{(\mu_{M-1}t_{M-1} + \mu_M \frac{n_{O(M-1)}^2}{n_{OM}^2})(\mu_{M-1}t_{M-1} + \mu_M)}{(1+t_{M-1}\frac{n_{O(M-1)}^2}{n_{OM}^2}\frac{\mu_M}{\mu_{M-1}})(1+t_{M-1}\frac{\mu_M}{\mu_{M-1}})} \right]$$

$$= \frac{(1-t_{M-1}^2)(\mu_{M-1}^2 - \mu_M^2 \frac{n_{O(M-1)}^2}{n_{OM}^2})}{(1+t_{M-1}\frac{n_{O(M-1)}^2}{n_{OM}^2}\frac{\mu_M}{\mu_{M-1}})(1+t_{M-1}\frac{\mu_M}{\mu_{M-1}})} \tag{117}$$

by elementary algebra. Now

$$\mu_{M-1}^2 - \mu_M^2 \frac{n_{O(M-1)}^2}{n_{OM}^2} = (\lambda^2 - n_{O(M-1)}^2 k^2) - (\lambda^2 \frac{n_{O(M-1)}^2}{n_{OM}^2} - n_{OM}^2 k^2 \frac{n_{O(M-1)}^2}{n_{OM}^2})$$

$$= \lambda^2 (1 - \frac{n_{O(M-1)}^2}{n_{OM}^2})$$

$$= N_{M(M-1)}\lambda^2. \tag{118}$$

Comparing them (118), (117) with (104) we see that (116) is valid when m = M - 1. Assume now (116) holds for m+1, i.e.

$$\mu_{m+1}^2 - U_{(m+1)z}U_{(m+1)h} = \lambda^2 \nu_{m+1}. \tag{119}$$

Then from (107) and (108)

$$\mu_m^2 - U_{mz}U_{mh} = \mu_m^2 - \left[\frac{(\mu_m t_m + \frac{n_{0m}^2}{n_{0(m+1)}^2} U_{(m+1)z})(\mu_m t_m + U_{(m+1)h})}{(1 + t_m \frac{n_{0m}^2}{n^2_{0(m+1)}} \frac{U_{(m+1)z}}{\mu_m})(1 + t_m \frac{U_{(m+1)z}}{\mu_m})}\right]$$

$$= \frac{(1 - t_m^2)\left[\mu_m^2 - \frac{n_{0M}^2}{n_{0(m+1)}^2} U_{(m+1)z}U_{(m+1)h}\right]}{(1 + t_m \frac{n_{0m}^2}{n_{0(m+1)}^2} \frac{U_{(m+1)z}}{\mu_m})(1 + t_m \frac{U_{(m+1)z}}{\mu_m})} \qquad (120)$$

Now using (119)

$$\mu_m^2 - \frac{n_{0m}^2}{n_{0(m+1)}^2} U_{(m+1)z}U_{(m+1)h}$$

$$= \mu_m^2 - \frac{n_{0m}^2}{n_{0(m+1)}^2}\left[(\mu_{m+1}^2 - \lambda^2 \nu_{m+1})\right]$$

$$= \mu_m^2 - \frac{n_{0m}^2}{n_{0(m+1)}^2} \mu_{m+1}^2 + \lambda^2 \frac{n_{0m}^2}{n_{0(m+1)}^2} \nu_{m+1}$$

$$= \lambda^2 \left[N_{(m+1)m} + \frac{n_{0m}^2}{n_{0(m+1)}^2} \nu_{m+1}\right] \qquad (121)$$

Comparing then (121), (120) with (109) we see that (116) holds for m.

3. DIPOLE FIELDS FOR THE HORIZONTALLY LAYERED MEDIA

3.1 Introduction

We shall in this Chapter use the general solution of Chapter 2 to obtain expressions for the electric field above the layered media when the source is an electric dipole located in the half-space above the layers. The assumed situation is illustrated in Figure 3.

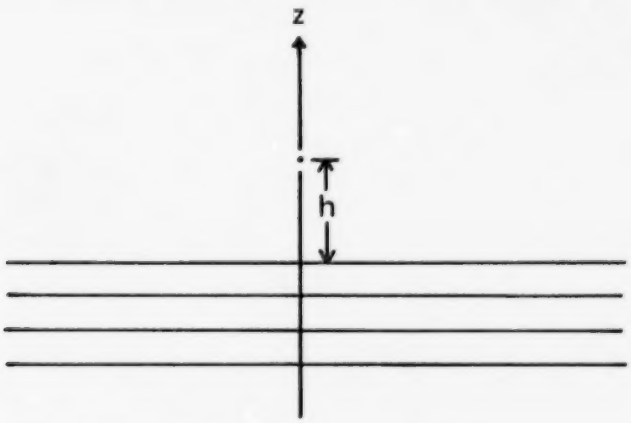

Figure 3 - DIPOLE SOURCE

It should be mentioned that although we are concentrating on the dipole, the results of Chapter 2 are valid for any finite antenna system, the dipole being the simplest of such systems.

We shall make use of (115) or its equivalent form

$$\underline{E}_x(z^+) = \underline{E}_{sx}(z^+) + e^{-z^+\mu_0}\left[\left(\frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\right)\left(\underline{E}_{osx} - \frac{jk_x\mu_0}{\lambda^2}\underline{E}_{osz}\right)\right.$$

$$\left. - \frac{jk_x\mu_0}{\lambda^2}\left(\frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\right)\underline{E}_{osz}\right)\Bigg]$$

$$\underline{E}_y(z^+) = \underline{E}_{sy}(z^+) + e^{-z^+\mu_0}\left[\left(\frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\right)\left(\underline{E}_{osy} - j\frac{k_y\mu_0}{\lambda^2}\underline{E}_{osz}\right)\right.$$

$$\left. - j\frac{k_y\mu_0}{\lambda^2}\,\frac{\mu_0 - \frac{1}{n_{01}^2}\,U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}}\,\underline{E}_{osz}\right] \qquad (122)$$

$$\underline{E}_z(z^+) = \underline{E}_{sz}(z^+) + e^{-z^+\mu_0}\left[\left(\frac{\mu_0 - \frac{1}{n_{01}^2}\,U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}}\right)\underline{E}_{osz}\right]$$

Equations (122) are derived from (115) by using the identity (116) with $m = 1$.

3.2 The Vertical Dipole

We now assume that the source current $\vec{J}_s$ in (12) is given by

$$\vec{J}_s = I\,dl\;\delta(x)\delta(y)\delta(z-h)\vec{a}_z \qquad (123)$$

Accordingly, the source field in (38) is given by

$$\vec{E}_s = T_{SE}(\vec{J}_s)*K_0 = \frac{1}{j\omega\varepsilon_0}\left[\nabla(\nabla.\vec{J}_s) + k^2\vec{J}_s\right] *K_0 \qquad (124)$$

Now

$$\nabla.\vec{J}_s = I\,dl\;\delta(x)\delta(y)\delta'(z-h) \qquad (125)$$

and

$$\nabla(\nabla.\vec{J}_s) = I\,dl[\delta'(x)\delta(y)\delta'(z-h)\vec{a}_x + \delta(x)\delta'(y)\delta'(z-h)\vec{a}_y$$

$$+ \delta(x)\delta(y)\delta''(z-h)\vec{a}_z] \qquad (126)$$

Therefore, the spatial (x and y) Fourier transform of $T_{SE}(\vec{J}_s)$ is

$$\underline{T_{SE}(\vec{J}_s)} = \frac{I\,d\ell}{j\omega\varepsilon_0}\ [(jk_x)\delta'(z-h)\vec{a}_x + (jk_y)\delta'(z-h)\vec{a}_y + (\delta''(z-h) + k^2)\vec{a}_z] \ . \qquad (127)$$

Also from (71)

$$\underline{K_0} = \frac{e^{-|z|\mu_0}}{2\mu_0} \ .$$

The source field has therefore the following components in the transformed (k_x, k_y) variables

$$\begin{aligned}
\underline{E}_{sx} &= \frac{I\ dl}{j\omega\varepsilon_0}\ (jk_x)\ \frac{\partial}{\partial z}\left[\left(\frac{e^{-|z-h|\mu_0}}{2\mu_0}\right)\right] \\
\underline{E}_{sy} &= \frac{I\ dl}{j\omega\varepsilon_0}\ (jk_y)\ \frac{\partial}{\partial z}\left[\left(\frac{e^{-|z-h|\mu_0}}{2\mu_0}\right)\right] \qquad (128) \\
\underline{E}_{sz} &= \frac{I\ dl}{j\omega\varepsilon_0}\ \left(\frac{\partial^2}{\partial z^2} + k^2\right)\left[\left(\frac{e^{-|z-h|\mu_0}}{2\mu_0}\right)\right].
\end{aligned}$$

Now referring to (84) and noting $z^- < 0 < h$, we have

$$\begin{aligned}
\underline{E}_{osx} &= \frac{I\ dl}{j\omega\varepsilon_0}\ (jk_x)\ \frac{e^{-h\mu_0}}{2} \\
\underline{E}_{osy} &= \frac{I\ dl}{j\omega\varepsilon_0}\ (jk_y)\ \frac{e^{-h\mu_0}}{2} \\
\underline{E}_{osz} &= \frac{I\ dl}{j\omega\varepsilon_0}\ (\mu_0^2 + k^2)\ \frac{e^{-h\mu_0}}{2} \qquad (129) \\
&= \frac{I\ dl}{j\omega\varepsilon_0}\ (\lambda^2)\ \frac{e^{-h\mu_0}}{2\mu_0}
\end{aligned}$$

Substituting (129) and (128) into (122) gives,

$$\underline{E}_x(z^+) = \frac{I\,dl}{j\omega\varepsilon_0}(jk_x)\frac{\partial}{\partial z^+}\left[\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{e^{-(z^+ + h)\mu_0}}{2\mu_0}\,\frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\right]$$

$$\underline{E}_y(z^+) = \frac{I\,dl}{j\omega\varepsilon_0}(jk_y)\frac{\partial}{\partial z^+}\left[\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{e^{-(z^+ + h)\mu_0}}{2\mu_0}\,\frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\right] \tag{130}$$

$$\underline{E}_z(z^+) = \frac{I\,dl}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z} + k^2\right)\left[\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{e^{-(z^+ + h)\mu_0}}{2\mu_0}\,\frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\right].$$

Taking then, the inverse Fourier transform of (130) yields,

$$E_x(z^+) = \frac{1}{j\omega\varepsilon_0}\frac{\partial^2}{\partial x\partial z}(\pi_z)$$

$$E_y(z^+) = \frac{1}{j\omega\varepsilon_0}\frac{\partial^2}{\partial y\partial z}(\pi_z) \tag{131}$$

$$E_z(z^+) = \frac{1}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z^2} + k^2\right)(\pi_z)$$

or more concisely,

$$\vec{E}(z^+) = T_{SE}(\pi_z \vec{a}_z) \tag{132}$$

where, the Fourier transform of π_z is given by

$$\underline{\pi}_z = I\,dl\left[\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{e^{-(z^+ + h)\mu_0}}{2\mu_0}\,\frac{\mu_0 - \frac{1}{n_{01}^2}U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}U_{1z}}\right]. \tag{133}$$

We may take the inverse transform of (133) as

$$\pi_z = \left[\frac{I\ dl}{4\pi}\ \frac{e^{-jkR_0}}{R_0} + \frac{e^{-jkR_1}}{R_1} - 2P\right] \tag{134}$$

where

$$R_0 = \sqrt{x^2+y^2+(z^+-h)^2} = \sqrt{\rho^2+(z^+-h)^2}$$

$$R_1 = \sqrt{x^2+y^2+(z^++h)^2} = \sqrt{\rho^2+(z^++h)^2}$$

$$P = \frac{1}{2\pi} \iint \frac{\frac{1}{n_{01}^2} U_{1h}\ e^{-(z^++h)\mu_0}}{\mu_0(\mu_0+U_{1z}\frac{1}{n_{01}^2})}\ e^{jk_x x+jk_y y}\ dk_x\ dk_y$$

$$= \int_0^\infty \frac{\lambda \frac{1}{n_{01}^2} U_{1z}\ e^{-(z^++h)\mu_0}}{\mu_0(\mu_0 + U_{1z}\frac{1}{n_{01}^2})} J_0(\lambda\rho)\ d\lambda$$

It may readily be seen that (132) agrees with the Sommerfeld (1909) flat earth theory by setting M = 1, i.e. a one layer structure. In this case

$$U_{1z} = U_{1h} = \mu_1 = \sqrt{\lambda^2-n_{01}^2k^2}\ . \tag{135}$$

It may also be verified that (132) agrees with the results of Wait (1970) for an M layer structure.

3.3 The Horizontal Dipole

In this case we assume that the source current $\vec{J}_s$ in (12) is given by

$$\vec{J}_s = I\ dl\ \delta(x)\ \delta(y)\ \delta(z-h)\vec{a}_x \tag{136}$$

i.e., a dipole aligned in the positive x-direction at coordinates (0, 0, h).

The two dimensional Fourier transform of $T_{SE}(\vec{J}_S)$ is then

$$\underline{T_{SE}(\vec{J}_s)} = \frac{I\,dl}{j\omega\varepsilon_0}\left[(k^2-k_x^2)\delta(z-h)\vec{a}_x + (jk_x)(jk_y)\delta(z-h)\vec{a}_y + jk_x\,\delta'(z-h)\vec{a}_z\right] \quad (137)$$

The source fields are, therefore,

$$\underline{E}_{sx} = \frac{I\,dl}{j\omega\varepsilon_0}(k^2 - k_x^2)\frac{e^{-|z-h|\mu_0}}{2\mu_0}$$

$$\underline{E}_{sy} = \frac{I\,dl}{j\omega\varepsilon_0}(jk_x)(jk_y)\frac{e^{-|z-h|\mu_0}}{2\mu_0} \quad (138)$$

$$\underline{E}_{sz} = \frac{I\,dl}{j\omega\varepsilon_0}(jk_x)\frac{\partial}{\partial z}\left(\frac{e^{-|z-h|\mu_0}}{2\mu_0}\right) .$$

The expressions corresponding to (129) are in this case

$$\underline{E}_{osx} = \frac{I\,dl}{j\omega\varepsilon_0}(k^2-k_x^2)\frac{e^{-h\mu_0}}{2\mu_0}$$

$$\underline{E}_{osy} = \frac{I\,dl}{j\omega\varepsilon_0}(jk_x)(jk_y)\frac{e^{-h\mu_0}}{2\mu_0} \quad (139)$$

$$\underline{E}_{osz} = \frac{I\,dl}{j\omega\varepsilon_0}(jk_x\mu_0)\frac{e^{-h\mu_0}}{2\mu_0} .$$

Therefore, substituting (139) into (115) yields

$$\underline{E}_x^{(z+)} = \frac{I\,dl}{j\omega\varepsilon_0}\left[(k^2-k_x^2)\left(\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\,\frac{e^{-(z^++h)\mu_0}}{2\mu_0}\right)\right.$$

$$\left. -jk_x\mu_0\,\frac{jk_x\left(N_{10} + \frac{1}{n_{01}^2}\,\nu_1\right)e^{-(z^++h)\mu_0}}{(\mu_0 + U_{1h})\left(\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}\right)}\right]$$

$$\underline{E}_y^{(z+)} = \frac{I\,dl}{j\omega\varepsilon_0}\left[(jk_x)(jk_y)\left(\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\,\frac{e^{-(z^++h)\mu_0}}{2\mu_0}\right)\right.$$

$$\left. -jk_y\mu_0\,\frac{jk_x\left(N_{10} + \frac{1}{n_{01}^2}\,\nu_1\right)\,e^{-(z^++h)\mu_0}}{(\mu_0 + U_{1h})\left(\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}\right)}\right] . \qquad (140)$$

$$\underline{E}_z^{(z+)} = \frac{I\,dl}{j\omega\varepsilon_0}\left[(jk_x)\,\frac{\partial}{\partial z}\left(\frac{e^{-|z^+-h|\mu_0}}{2\mu_0}\right)\right.$$

$$\left. + jk_x\mu_0\,\frac{\mu_0 - \frac{1}{n_{01}^2}\,U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}}\,\frac{e^{-(z^++h)\mu_0}}{2\mu_0}\right] .$$

Now in the expression for $\underline{E}_z^{(z^+)}$ above the term

$$\frac{\mu_0 - \frac{1}{n_{01}^2}\,U_{1z}}{\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}} = \frac{2\lambda^2\left(N_{10} + \frac{1}{n_{01}^2}\,\nu_1\right)}{(\mu_0 + U_{1h})\left(\mu_0 + \frac{1}{n_{01}^2}\,U_{1z}\right)} - \frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}} \quad . \qquad (141)$$

This follows from (116). Therefore, the z-component of (140) may also be written as,

$$\underline{E}_z^{(z^+)} = \frac{I\,dl}{j\omega\varepsilon_0}\left[(jk_x)\frac{\partial}{\partial z}\left(\frac{e^{-|z^+-h|\mu_0}}{2\mu_0}\right) - (jk_x\mu_0)\frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\frac{e^{-(z^++h)}}{2\mu_0}\right.$$

$$\left. + \lambda^2 \frac{jk_x\left(N_{10} + \frac{1}{n_{01}^2}\nu_1\right)e^{-(z^++h)}}{(\mu_0 + U_{1h})\left(\mu_0 + \frac{1}{n_{01}^2}U_{1z}\right)}\right] . \qquad (142)$$

Taking inverse Fourier transforms of (140), (142) gives

$$\vec{E} = T_{SE}(\pi_x\vec{a}_x + \pi_z\vec{a}_z) \qquad (143)$$

where π_x and π_z are inverse transforms of

$$\underline{\pi}_x = I\,dl\left[\frac{e^{-|z^+-h|\mu_0}}{2\mu_0} + \frac{\mu_0 - U_{1h}}{\mu_0 + U_{1h}}\frac{e^{-(z^++h)\mu_0}}{2\mu_0}\right] \qquad (144)$$

$$\underline{\pi}_z = I\,dl\left[\frac{jk_x\left(N_{10} + \frac{1}{n_{01}^2}\nu_1\right)e^{-(z^++h)\mu_0}}{(\mu_0 + U_{1h})\left(\mu_0 + \frac{1}{n_{01}^2}U_{1z}\right)}\right] . \qquad (145)$$

Equation (144) may be partially inverted as

$$\pi_x = \frac{I\,dl}{4\pi}\left[\frac{e^{-jkR_0}}{R_0} - \frac{e^{-jkR_1}}{R_1} + 2H\right] \qquad (146)$$

where

$$H = \frac{1}{2\pi}\iint \frac{e^{-(z^++h)\mu_0}}{\mu_0 + U_{1h}} e^{jk_x + jk_y y}\,dk_x dk_y$$

$$= \int_0^\infty \frac{\lambda e^{-(z^++h)\mu_0}}{\mu_0 + U_{1h}} J_0(\lambda\rho)\,d\lambda$$

$$R_0 = \sqrt{x^2+y^2+(z^+-h)^2} = \sqrt{\rho^2+(z^+-h)^2}$$

$$R_1 = \sqrt{x^2+y^2+(z^++h)^2} = \sqrt{\rho^2+(z^++h)^2}$$

In the case of flat earth (1 layer), we have

$$U_{1h} = U_{1z} = \mu_1$$
$$\nu_1 = 0 \tag{147}$$

and so

$$H = \int_0^\infty \frac{\lambda e^{-(z^+ + h)\mu_0}}{\mu_0 + \mu_1} J_0(\lambda\rho)\, d\lambda.$$

Also, from (145)

$$\pi_z = \frac{I\, dl}{4\pi^2} \iint \frac{jk_x\left(1 - \frac{1}{n_{01}^2}\right) e^{-(z^+ + h)\mu_0}}{(\mu_0 + \mu_1)\left(\mu_0 + \frac{1}{n_{01}^2}\mu_1\right)} e^{jk_x x + jk_y y}\, dk_x dk_y$$

$$= \frac{I\, dl}{2\pi k^2} \frac{\partial}{\partial x} \int_0^\infty \frac{\lambda(\mu_0 - \mu_1) e^{-(z^+ + h)\mu_0}}{n_{01}^2\mu_0 + \mu_1} J_0(\lambda\rho)\, d\lambda \tag{148}$$

$$= -\frac{I\, dl}{2\pi k^2} \cos\phi \int_0^\infty \frac{\lambda^2(\mu_0 - \mu_1)}{n^2{}_{01}\mu_0 + \mu_1} e^{-(z^+ + h)\mu_0} J_1(\lambda\rho)\, d\lambda$$

where

$$\cos\phi = \frac{x}{\rho}.$$

It is easily verified that this agrees with classical Sommerfeld (1949) flat earth theory.

3.4 Approximation of the Fields for the Vertical Dipole

In this section we shall discuss an approximation of the integral in (134) and the resulting electric fields above the interface. For reference, we require an approximation of

$$P = \int_0^\infty \frac{\lambda \, 1/\eta_{01}^2 \, U_{1z} \, e^{-(z^+ + h)\mu_0}}{\mu_0(\mu_0 + 1/\eta_{01}^2 \, U_{1z})} J_0(\lambda\rho) \, d\lambda \tag{149}$$

The integral P above has been a topic of discussion for many years. The interested reader is referred to Wait (1970) for details. The notation in (149) has been chosen to agree with Wait. In order to complete the agreement we put

$$jk\Delta = 1/\eta_{01}^2 \, U_{1z}$$

or

$$\Delta = \frac{1}{jk\eta_{01}^2} U_{1z} \quad . \tag{150}$$

Then (149) becomes

$$P = \int_0^\infty \frac{(jk\Delta) \, \lambda \, e^{-(z^+ + h)\mu_0}}{\mu_0 \, (\mu_0 + jk\Delta)} J_0(\lambda\rho) \, d\lambda \tag{151}$$

which is the integral discussed by Wait. P can be asymptotically evaluated using a saddle point technique under the following assumptions on Δ:

1) Δ is a constant, i.e. independent of λ

2) $|\Delta| << 1$

The first of these assumptions is, of course, not strictly true, for Δ is a function of λ. However, it may be observed (see (99)) that the variation of Δ with respect to λ is the result of expressions of the form

$$\sqrt{\mu_m} = \lambda^2 - k^2 \eta_{0m}^2$$

where k is the free space wave number, and η_{0m} the refractive index in the m^{th} layer. Now if $|\eta_{0m}^2| >> 1$ and we agree that the "important" part of the integral is for $0 < \lambda < k$, we can put

$$\mu_m \sim jkn_{0m} \tag{152}$$

Another approximation, suggested by Wait, is to put λ equal to its value at the saddle point of the integral. In this case, the value is

$$\lambda = k \left(\frac{\rho}{R_1}\right) \tag{153}$$

where ρ and R_1 are defined in (134).

Under the assumptions discussed above, the integral P may be evaluated as,

$$P \sim j(\pi p_e)^{1/2} e^{-w} \operatorname{erfc}(jw^{1/2}) \frac{e^{-jkR_1}}{R_1} \tag{154}$$

where

$$w = p_e\left(1 + \frac{z^+ + h}{\Delta R_1}\right)^2 \tag{155}$$

$$p_e = -j \frac{kR_1}{2} \Delta^2 \tag{156}$$

and erfc is the complementary error function. A form which is convenient for computation may be found by noting that

$$1 + \frac{z^+ + h}{\Delta R_1} = 1 + \frac{\sin \theta}{\Delta} \tag{157}$$

where θ is the angle defined in Figure 4.

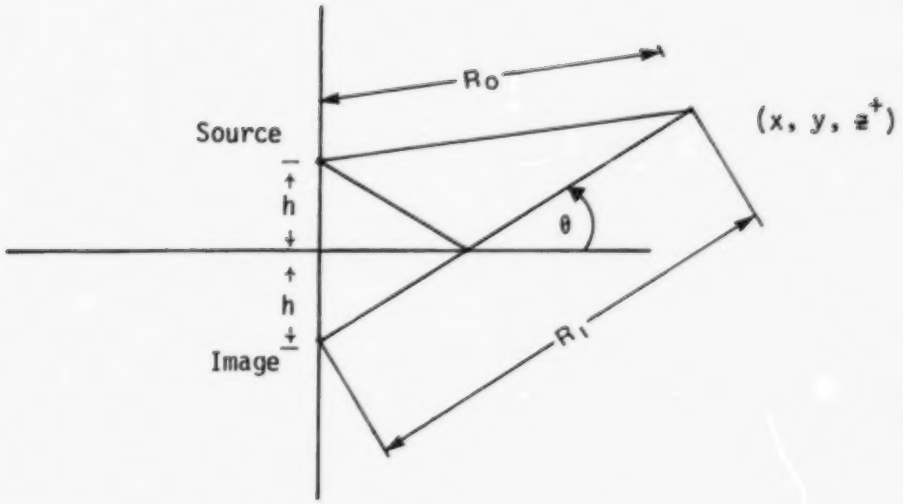

Figure 4 - Illustration for (163) and (164)

We also define

$$R_\nu = \frac{\sin\theta - \Delta}{\sin\theta - \Delta} \tag{158}$$

Then (155) becomes

$$w = p_e \left(\frac{2}{1 - R_\nu}\right)^2 \tag{159}$$

and (154)

$$P \sim j \frac{(1 - R_\nu)}{2} (\pi w)^{1/2} e^{-w} \operatorname{erfc}(jw^{1/2}) \frac{e^{-jkR_1}}{R_1} \tag{160}$$

Therefore, the expression for π_z from equation (134) may be written as,

$$\pi_z = \frac{I\, dl}{4\pi} \left[\frac{e^{-jkR_0}}{R_0} + R_\nu \frac{e^{-jkR_1}}{R_1} + (1 - R_\nu) F(w) \frac{e^{-jkR_1}}{R_1}\right] \tag{161}$$

where

$$F(w) = 1 - j(\pi w)^{1/2} e^{-w} \operatorname{erfc}(jw^{1/2}) \tag{162}$$

The electric field is now easily found from (132). Noting that π_z is not angular dependent, we have

$$E_z = \frac{k^2 I\, dl}{j\omega\varepsilon_0(4\pi)} \cos^2\theta \left[\frac{e^{-jkR_0}}{R_0} + R_\nu \frac{e^{-jkR_1}}{R_1} + (1-R_\nu)F(w) \frac{e^{-jkR_1}}{R_1}\right] \tag{163}$$

$$E_\rho = -\frac{k^2 I\, dl}{j\omega\varepsilon_0(4\pi)} \left[\sin\phi \cos\phi \frac{e^{-jkR_0}}{R_0} + \sin\theta \cos\theta\, R_\nu \frac{e^{-jkR_1}}{R_1}\right. \tag{164}$$

$$\left. - \cos\theta\,(1 - R_\nu)\Delta F(w) \frac{e^{-jkR_1}}{R_1}\right]$$

where

$$\sin\phi = \frac{z^+ - h}{R_0}\,, \quad \cos\phi = \frac{\rho}{R_0}\,.$$

Far field approximations have been used in (163) and (164).

We shall use (163) and (164) in the discussions of the next chapter. It should be noted, however, that more accurate estimates of (149) are available, for example, Furutsu (1959).

4 CONCLUSIONS

4.1 Discussion

The solutions, as developed and presented in Chapter 2, are general formal solutions to the problem of electromagnetic interaction with horizontally layered isotropic media as posed. The analytic technique is "different" from that which is standard in the literature. In general, this method of analysis appears to be more powerful, from a formulation point of view, than more traditional methods. It has the following properties: the source or exciting field (antenna) is arbitrary, although certainly a realization condition is implied, i.e., the current source is assumed to be spatially bounded; a solution for the resulting electric field intensity is found directly as a function of the exciting electric field intensity, rather than through Hertz vectors; the boundary conditions are given directly by auxiliary equations which evolve directly from the decomposition technique. It should be noted that the method has been employed successfully for the formulation of the problem of electromagnetic interaction with rough surfaces (Walsh, 1980), and currently is being used successfully in the problem of "mixed path" propagation and scatter.

The problem of solving equations remains. In the case of rough surfaces integral equations must be solved. In the present case algebraic equations must be solved. In the author's opinion this need will always be present when dealing with physical problems. The formulation method however, can affect the complexity of the equations, and bring out their relevant properties.

The problem of inverting two dimensional Fourier transforms also remains, with the resulting asymptotic expansions using saddle point integration etc. This can be avoided if one does not use Fourier transforms or "resolution into plane waves". The price however, must then be paid in equation complexity and usually approximations of another kind must be employed (for example, Hufford (1952).

The expressions for the electric field intensities from dipole sources as contained in Chapter 3 are largely classical (Wait, 1970). Indeed, the solutions have been manipulated to show the relationship with classical theory. The full potential of the general solution of Chapter 2 has not been exploited, and this should form the basis of a future study.

A word on plane waves is in order. Within the recursive calculations of Chapter 2 plane wave solutions are always in closed form. This is obvious when it is realized that the spatial Fourier transform of plane wave sources are Dirac delta functions. The various inversions are then trivial (see Walsh, 1980, for a discussion). Equation (87) and (97) are plane wave solutions, hence the term "resolution into plane waves". Historically, resolution into plane waves is a step in the evolution of modern Fourier techniques.

4.2 Suggested Model for Propagation Predictions Over Sea Ice

It was originally intended to include in this report a chapter of calculations for models of propagation over sea ice. However, a ground wave propagation experiment was carried out by C-CORE in March 1982 at Pond Inlet. In view of this it was decided to include the various computer programs and calculations with the experimental results under

the same cover. A separate report will be forthcoming later in 1983.

The basic idea is to model the ice-water halfspace as a layered medium and to use the results of Chapter 3, i.e., equation (163) for electric field intensity calculations. Equation (163) is of course a "flat earth" model. If necessary, earth curvature effects can be introduced by using values of surface impedance calculated from equation (150) in a standard spherical earth computer program (e.g., Berry, 1978). This technique has already been used by Hill and Wait (1981). In any event, a comparison with measurement is being made in the forthcoming report and conclusions drawn as to the adequacy of the model.

REFERENCES

Berry, L.A. 1978, User's guide to low-frequency radar coverage programs, U.S. Department of Commerce, OT Technical Memorandum 78-247, 150 pp.

Furutsu, K. 1959, On the excitation of waves of proper solution, Trans. I.R.E., Special Supplement, AP-7, pp. 209-218.

Hill, D.A., and Wait, J.R. 1981, HF ground wave propagation over sea ice for a spherical earth model, IEEE Transactions on Antennas and Propagation, Vol. AP-29, No. 3, May, pp. 525-527.

Hill, D.A., and Wait, J.R. 1981, HF radio wave transmission over sea ice and remote sensing possibilities, IEEE transactions on Geoscience and Remote Sensing, Vol. GE-19, No. 4, October, pp. 204-209.

Hufford, G.A. 1952, An integral equation approach to the problem of wave propagation over an irregular surface, Quart. Appl. Math, 9, pp 391-404.

Sommerfeld, A. 1949, Partial Differential Equations, Academic Press, New York.

Wait, J.R. 1970, Electromagnetic Waves in Stratified Media, Pergamon, Manwell House, Fairview Park, Elmsford N.Y.

Wait, J.R. 1980, Electromagnetic surface impedance for a layered earth for general excitation, Radio Science, Vol. 15, No. 1, pp. 129-134.

Walsh, J. 1980, On the theory of electromagnetic propagation across a rough surface and calculation in the VHF, region, Technical report prepared for the Defence Research Establishment Atlantic (DREA) 200 pp.

This list of references is not exhaustive. For further information see Wait (1970)